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# On a predictive macroscopic contact-sliding wear model based on micromechanical considerations

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**Abstract**

A model for the mild wear of two contacting solids and an analytical example are proposed in this article. The model includes the presence of an interface made of damaged materials, fluid and wear debris. It consists in a wear criterion, an interface law and complementary relations deduced from the mass conservation. A thermodynamical analysis provides energy-release rates associated with the evolution of the surfaces in contact and the mass fluxes due to wear. They are used as characteristic quantities in the formulation of the wear criterion and wear velocities. Given that the physics of the interface modify the global contact conditions, micromechanical considerations are developed and result in an interface law, modeling its evolution with an internal parameter: the volume fraction of wear debris. The relation between this parameter and wear velocities is obtained with the mass conservation equation, which completes the model and allows to apply it in a numerical simulation. As an example, a problem of a rigid punch sliding on an elastic worn-out half-plane is treated by means of integral equations, accounting on the presence of an interface according to the previous modeling. Stresses and strains are obtained analytically, as asymptotic expansion fields. 

*Keywords:* Friction; Wear; Energy release rate; Third body; Interface; Integral equations; Asymptotic expansion

## 1. Introduction

Although contact-wear phenomena are of frequent occurrence, many difficulties appear when engineers attempt to control them. Friction between contacting surfaces induces damage of materials, producing surface and subsurface cracks. As wear occurs, asperities of damaged surfaces are cracked, leading to loss of material and debris appearance in the contact interface. By the way, as wear occurs, geometrical changes take place and contact conditions are significantly modified because of the presence of wear particles. Life expectancy of machines can be reduced seriously, which implies the need for specific controls. This phenomenon is observed in nuclear power plants where security must be ensured, despite the wear of some components, therefore implying these components to be changed frequently. For complex structures, a

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solution to this problem consists in using numerical simulations as a means to predict damage and to select the most resistant materials. This raises the question of a predictive model for contact wear, which could enable the evaluation of damage and loss of material by numerical calculations.

However, during the contact wear process, several mechanisms occur, either simultaneously or not, making an attempt to analyze very difficult. Loss of material produced by asperities cracks can be increased by corrosion; particles can be abrasive, or on the contrary, small enough to cause few effects in the interface. Ko (1997) precisely describes the different wear mechanisms and provides an analysis of the wear of components in nuclear power plants. His article confirms the complexity of wear problems, making their modeling difficult under conditions considered. Several models are presented in (Ko, 1997). An analysis of more than 5000 papers published by "Wear" is proposed by Meng and Ludema (1995). In the literature, mostly empirical equations are proposed, established for a particular system using specific parameters. These works provide informations about specific wear mechanisms and processes, but cannot be applied to other experimental conditions, which means that related equations are no more available. Some authors try to compare mechanical properties of materials, like fatigue limit, with their resistance to wear. Many studies are based on the Archard's equation (Archard, 1953) giving the volume of lost material as proportional to the normal load, to the sliding distance, the coefficient of proportionality being called wear coefficient. This one depends on operating conditions, therefore experimental data are needed for reliable predictive simulations. Strömberg et al. (1996) developed a comprehensive generalized standard model for wear, where Archard's law is introduced and its coefficient slightly modified. Few analytical models exist; one can refer to Zmitrowicz (1987a,b,c) for a complete mathematical framework.

More generally, global existing models do not take account of the physics of the interface, being unable, by the way, to provide fair results when different mechanisms can occur. As written in (Singer and Wahl, 1999), macroscopic modeling should not ignore the influence of the interface. Moreover, the complexity of a complete micromechanical model discourages its use for macroscopic modeling. Nevertheless, as pointed out by Meng and Ludema (1995), the translation from microscopic observation to macroscopic laws may be done anyway, as a necessity for a predictive wear model.

The aim of this work is to develop a macroscopic model for mild wear, which includes interface related parameters, complex enough to describe the influence of the preponderant physical phenomena, and simple enough to allow numerical simulations. A basic system of two contacting bodies with an interfacial fluid is studied. The fluid can be a lubricant or not, compressible or not. The loss of material is characteristic of the wear of one or both solids.

In this paper, a thermodynamical analysis of this system is advanced. Conservation laws are written taking into account the mass fluxes and the moving boundaries due to wear. From this, energy-release rates are obtained, depending on the stress and strain state in the sound material and the damaged one. A wear criterion for each solid is formulated, based on the energy-release rates. The interface contains the damaged parts of both materials and the third-body (fluid and debris). An interface model is necessary to determine the quantities involved in the wear criterion and to calculate the mechanical state of the tribological system.

This macroscopic model is presented in Section 2. In Section 3, this model is built on micromechanical considerations, including an internal variable (volume fraction of wear particles in the interface). It may be interesting to compare the interface law inferred from this study with the Ruina–Kirchhoff friction law (Scholz, 1998), also expressed with an internal evolutive parameter. The mass conservation provides a relation between the internal variable and the wear criterion presented in Section 2.

These complementary studies allow a wear simulation. The problem of a rigid punch sliding on an elastic half plane submitted to wear and their interface is proposed in Section 4, as an example, treated analytically with integral equations. It may be noted that generally, both contacting solids are not simultaneously losing material; consequently, assuming that the punch is not worn-out is not an important restriction. The problem's statement results in one essential equation (called the wear equation) to solve. Assuming the fluid incompressible, asymptotic expansion solutions with respect to the small parameter (fraction of wear particles in the interface) are searched, which allows analytical treatment of the wear equation. The zero-order solution without wear is given; the first-order solution with wear of the elastic half plane can then be determined.

## 2. Macroscopic modeling of contact wear phenomena

In this section, a thermodynamical approach of a system of two contacting solids is developed, in the case of mild wear of one or both solids and loss of material. The interfacial layer of the bodies is more precisely described in the next section; in this one, a global model is built with the assumption of mass fluxes from the worn solids to the interface, non-zero fluxes being characteristic of wear phenomena. Similar assumptions are made in a complex mathematical framework for wear of two contacting solids proposed by Zmitrowicz (1987a,b,c); however, his study needs further physical interpretation. Before developing the thermodynamical approach, let us consider Stribeck's curves in order to distinguish mild wear from severe wear mechanisms, and to infer some evidence for our study.

### *2.1. Stribeck's curve: regimes I–III*

Wear experiments provide $x$–$y$ plots with $x = \eta V/p$ and $y$, the friction coefficient ($\eta$: fluid–lubricant viscosity, $V$: relative velocity of the contacting bodies, $p$: pressure) called Stribeck's curves, schematically represented in Fig. 1. In the case where $\eta V/p$ is small, it may be noted that the friction coefficient is high (I), which corresponds to dry and severe wear, when surfaces are not entirely protected by the fluid. As $\eta V/p$ increases, the fluid is spread over the contacting surfaces and the friction coefficient decreases. An unstable phase (II) follows, with an incomplete fluid cover: here combined dry and lubricated wear mechanisms occur. Phase (III) corresponds to a stable lubricated hydrodynamic regime, where an interface made of fluid is formed. Havet (1998) presents some tribological aspects of lubrication and Stribeck's curves, giving references on this topic.

This confirms the interaction between the so-called first bodies (contacting solids) and the third body (fluid and wear products). This evidence results in the necessity to associate a global approach to the wear modeling with the description of the interface evolution, through some internal state variable as will be proposed later. Our purpose in this section is to propose a macroscopic model for the wear of two bodies in

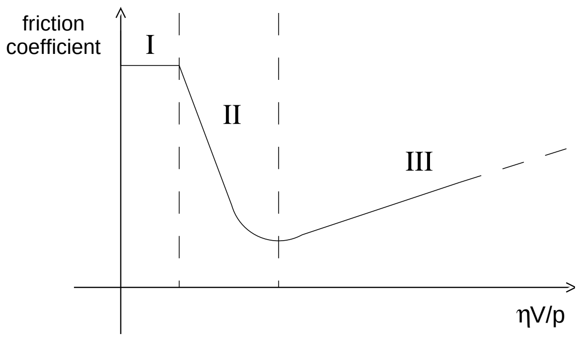


Fig. 1. Stribeck's curve.

contact, the interface being the subject of principal interest in Section 3. Both complementary theoretical approaches, thermodynamical analysis and interface law, can then be used for numerical simulations of the wear of a structure.

## 2.2. The energy release rate approach in wear mechanics

In the present analysis, assuming that wear produces a characteristic dissipation, energy release rates for the phenomena are determined. Energy release rates are used in fracture mechanics to describe cracks propagation. Here, these quantities characterize the production of wear debris by damage of solid and particle detachment. They are used in the formulation of a wear criterion, which is completed in the further study by the interfacial film behavior.

### 2.2.1. Three-domain model

Let us consider a system of two solids in contact in Fig. 2, one moving with respect to the other. Apparently smooth surfaces are, on a large scale, made of asperities; subsurface materials instead of being unaffected, are damaged, cracked because of pressure and friction due to the movement. $\Gamma_1$ and $\Gamma_2$ are boundaries separating sound materials from the damaged ones. In fact, the density of cracks is increasing continuously if we move to the contacting edge of solids 1 and 2. In this model, we will distinguish sound material from the damaged one approximately and separate them by the boundary $\Gamma_i$, considering that material becomes damaged and goes through $\Gamma_i$, once its macroscopic behavior is no more known and controlled. By the way, we build a three-domain model (third picture in Fig. 2): $\Omega_1$ and $\Omega_2$ are, respectively, the sound parts of the solids 1 and 2, $\Omega_3$ is the interface with a complex behavior containing damaged parts of both solids and the third body made of wear products mixed with the fluid, eventually the lubricant.

When the solid $i$ is worn, particles in $\Omega_3$ are detached and cracks propagate towards sound material; there are mass fluxes through $\Gamma_i$ from $\Omega_i$ to $\Omega_3$. (The fluid and debris flow in $\Omega_3$ is described in the next section.) Balance equations are written for the system $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$ with moving boundaries $\Gamma_1$ and $\Gamma_2$. See Pradeilles-Duval and Stolz (1995) for the study of a system with moving boundaries, and Dragon-Louiset and Stolz (1999) for details about this analysis.

### 2.2.2. Conservation laws and entropy production

Let us introduce the jump or discontinuity of a quantity $b$ through the moving surface $\Gamma_i$, $i = 1, 2$: $[\![b]\!]_i^3 = b^{3i} - b^i$ where $b^i$ defines the value of $b$ on the $\Omega_i$ side of $\Gamma_i$, and $b^{3i}$ defines its value on the $\Omega_3$ side of

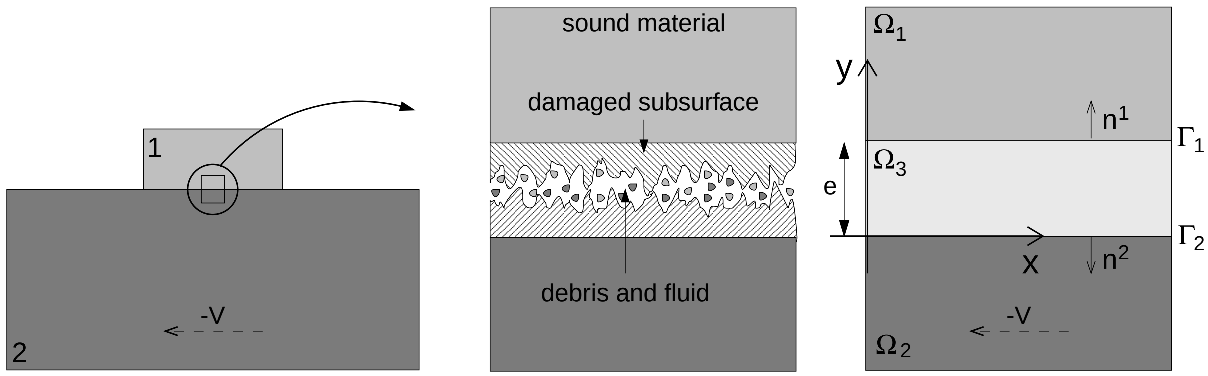


Fig. 2. Three-domain model.

$\Gamma_i$. $W^i$ is the geometrical celerity of $\Gamma_i$, $n^i$ is its unit normal vector as shown in Fig. 2. In Eulerian formulation, $U$ defining the velocity, time derivative of integrals of $b$ over the domain $\Omega$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega b\,\mathrm{d}\Omega = \sum_{i=1,2,3}\int_{\Omega_i}\left[\frac{\partial b^i}{\partial t} + \mathrm{div}(b^i U^i)\right]\mathrm{d}\Omega + \sum_{i=1,2}\int_{\Gamma_i} [\![ b(U - W^i) ]\!]_i^3 n^i\,\mathrm{d}\Gamma. \tag{1}$$

Integrals over $\Gamma_1$ and $\Gamma_2$ come therefore from the conservation laws applied to $\Omega$. Conservation equations are obtained for $\Omega_i$ $(i = 1, 2, 3)$ and $\Gamma_i$ $(i = 1, 2)$. Mass conservation gives the mass fluxes $\mu^i = \rho^i(U^i - W^i)n^i = -\rho^i v^i$ $(i = 1, 2)$, with $v^i = (W^i - U^i)n^i$ the material velocity of the boundary $\Gamma_i$. The equation, $\mu^i[\![\psi + sT]\!]_i^3 = [\![U]\!]_i^3[(\sigma^{3i} + \sigma^i)/2]n^i - [\![q]\!]_i^3 n^i$ $(i = 1, 2)$, arises from the law of the conservation of energy ($\psi$ denotes the free energy, $s$ the entropy, $T$ the temperature and $q$ the heat flux). According to the second law of thermodynamics, the internal entropy production $\mathsf{W}$ is non-negative; for both boundaries $\Gamma_1$, $\Gamma_2$: $\mathsf{W}_{\Gamma_i} = \mu^i[\![s]\!]_i^3 + [\![q/T]\!]_i^3 \geqslant 0$, and in $\Omega_3$, $\mathsf{W}_{\Omega_3} = \int_0^{e(x)}[\sigma : \dot{\varepsilon} - \rho(\dot{\psi} + s\dot{T}) - q(\nabla T/T)](1/T)\,\mathrm{d}y \geqslant 0$ ($e(x)$ is the thickness of the layer). We suppose from now the continuity of the temperature $T$ through both boundaries ($T = T^i$ on $\Gamma_i$).

### 2.2.3. Wear criterion

The entropy production due to the wear of $\Omega_i$ was expressed above as $\mathsf{W}_{\Gamma_i}$, which can be transformed with the equation of conservation of energy and simplified thanks to two assumptions. At first, on the assumption that wear is mild, we presume that normal stresses are continuous through $\Gamma_i$. However, owing to the balance equation of momentum: $[\![\sigma]\!]_i^3 n^i = \mu^i[\![U]\!]_i^3$, $(i = 1, 2)$; which presupposes that $\mu^i[\![U]\!]_i^3$ is a second-order term. Secondly, displacements $\xi$ through $\Gamma_i$ are supposed to be continuous. Hadamard consistency relations provide $[\![U]\!]_i^3 = -v^i[\![\nabla\xi]\!]_i^3 n^i$ $(i = 1, 2)$. This enables us to write $\mathsf{W}_{\Gamma_i}$ in the following form:

$$\mathsf{W}_{\Gamma_i} = \frac{v^i}{T^i}(g^i - g^{3i}) \quad \text{with} \begin{cases} g^i = n^i\sigma^i\nabla\xi^i n^i - \rho^i\psi^i, \\ g^{3i} = n^i\sigma^{3i}\nabla\xi^{3i}n^i - \rho^i\psi^{3i}, \end{cases} \tag{2}$$

where $g^i$ and $g^{3i}$ are energy release rates associated to the wear of solid $i$. Assuming that there exists a wear threshold $g^{is}$ defined for the material of the solid $i$, we formulate a wear criterion as follows:

$$\begin{cases} \text{if } G(g^i, g^{3i}) = g^i - g^{3i} - g^{is} < 0, & \text{no wear occurs,} \\ \text{if } G(g^i, g^{3i}) = g^i - g^{3i} - g^{is} \geqslant 0, & \text{wear of } i \text{ occurs.} \end{cases} \tag{3}$$

The velocity $v^i$ can be inferred from the criterion where wear occurs: $v^i = F(\langle G(g^i, g^{3i})\rangle_+)$ where $F$ is a function ($\langle b\rangle_+$ is the positive part of $b$, i.e. $\langle b\rangle_+ = 0$ if $b \leqslant 0$, $\langle b\rangle_+ = b$ if $b > 0$). Several forms of $F$ can be examined (and confronted with experiments), e.g. linear ($v^i = \beta\langle G(g^i, g^{3i})\rangle_+$), or polynomial. Another way to determine $v^i$ can be to develop a normality law, similar to plasticity; this will be studied in the future.

Although $g^i$ is easy to determine in the case where both contacting solids are elastic (it seems sufficient for mild wear process), $g^{3i}$ needs further study. The influence of the interface on the first bodies is expressed by $g^{3i}$; it contains the physics of $\Omega_3$. We attempt to approach the behavior of the interface in the following section.

## 3. A multiscale model of the third body

### 3.1. The three-scale third body

Our aim is not to describe and model the microscopic mechanisms which occur in $\Omega_3$, but to use micromechanical considerations in order to describe in a realistic manner the $\Omega_3$ physics and then model it on a mesoscopic scale. A relevant interface law is inferred from this analysis, for applications on a macroscopic

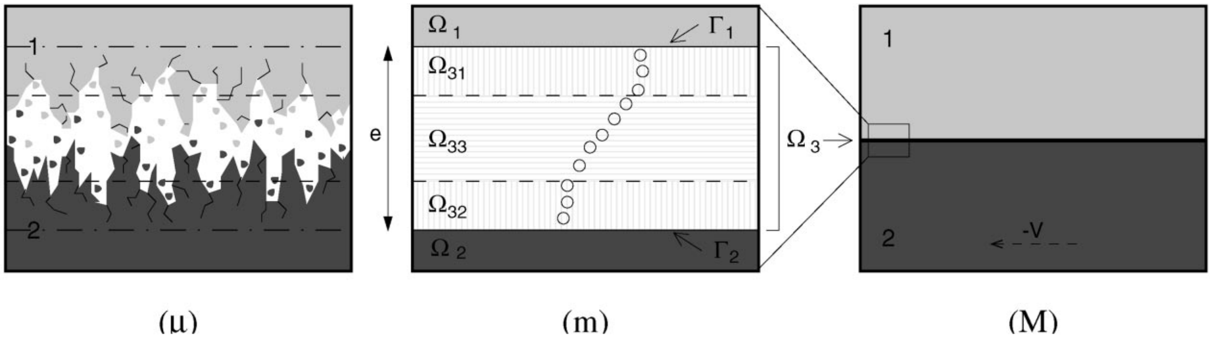


Fig. 3. From the microscopic scale to the macroscopic one.

scale. Consider the Fig. 3 where three scales from the microscopic to the macroscopic one via a mesoscopic scale are represented:

($\mu$), *microscopic scale:* Asperities in the third body on microscopic scale, wear mechanisms. Some authors propose studies of plastic strains, cracks propagation, contact between two asperities on the microscopic scale (Barbarin, 1997; Stupkiewicz and Mróz, 1999).

- $\Omega_{31}$, $\Omega_{32}$: damaged material, cracks, wear particles jammed into the holes between asperities. The fluid does not soak completely into the cracks;
- $\Omega_{33}$: contact between asperities which are progressively detached. The fluid carrying the debris along.

($m$), *mesoscopic model:* Third body on mesoscopic scale: $\Omega_3 = \Omega_{31} \cup \Omega_{33} \cup \Omega_{32}$. This three-area model was proposed by Godet (1982; 1990) and completed by Georges (1999), Georges et al. (1993), Berthier (1989; 1990), Berthier et al. (1988).

- $\Omega_{31}$, $\Omega_{32}$: porous, damaged media with no shear, no shear-stress but only strains $\varepsilon_{xx}$, $\varepsilon_{yy}$ and stresses $\sigma_{xx}$, $\sigma_{yy}$;
- $\Omega_{33}$: solid particle suspension forming a sheared layer and viscous fluid flow (Dragon-Louiset, 2000). Strains $\varepsilon_{xy}$ and $\varepsilon_{yy}$, shear-stress $\sigma_{xy}$ and compressive stress $\sigma_{yy}$.

($M$), *macroscopic model:* Interface on the macroscopic scale: $\Gamma = \Gamma_1 \cup \Omega_3 \cup \Gamma_2$. There are several models and friction laws in the literature with an internal variable, for example the Ruina-Kirchoff's friction law (Scholz, 1998). In this paper, we will propose an interface law adapted to the wear phenomena, giving a relation between the compressive stress $\sigma_{yy}$ and the strain $\varepsilon_{yy}$, the shear-stress $\sigma_{xy}$ and $\dot{\varepsilon}_{xy}$ (because of the viscosity of the solid particle suspension). $\sigma_{yy}$, $\varepsilon_{yy}$, $\varepsilon_{xy}$, and $\varepsilon_{xx}$ are related to the boundary conditions on $\Gamma_1$ and $\Gamma_2$ for a macroscopic point of view. Note: the friction coefficient can be inferred from the interface behavior by $\mu = \overline{\sigma}_{xy}/\overline{\sigma}_{yy}$ where overlined fields mean averages on the layer thickness.

### 3.2. Macroscopic interface law

We will assume, for the application, steady-state wear, 2D, plane strain, incompressible fluid, 2 moves at the speed of $-V\underline{x}$ with respect to 1, $x$ and $y$ axis are fixed in 1.

#### 3.2.1. Internal variable for the layer $\Omega_3$

Internal variable: $\varphi$ volume fraction of the particles ($\varphi = \varphi^1 + \varphi^2$ particles from $\Omega_1$ and $\Omega_2$) then $\varphi^{\mathrm{f}}$ is the volume fraction of fluid and $\varphi^1 + \varphi^2 + \varphi^{\mathrm{f}} = 1$. Relations between $\varphi$, $e$, $v$ wear velocity and $V$ the relative velocity between 1 and 2 are given by the conservation of mass for the three types of material (debris of 1, debris of 2 and fluid).

The balance of mass is $(\partial\rho^3/\partial t) + \mathrm{div}(\rho^3 U^3) = 0$ where $\rho^3 = \rho^1\varphi^1 + \rho^2\varphi^2 + \rho^f\varphi^f$. In the steady-state case, $e = e(x,y)$, $\varphi^i = \varphi^i(x,y)$ for $i = 1,2,\mathrm{f}$. Mean values of volume fractions can also be defined by: $\overline{\varphi}^i(x) = (1/e(x))\int_0^{e(x)} \varphi^i(x,y)\mathrm{d}y$ for $i = 1,2,\mathrm{f}$.

As we described $\Omega_3$, $\Omega_{3i}$, $i = 1,2$ are the intermediate areas for the material, which is firstly damaged during the process, goes through $\Gamma_i$ at the wear velocity $v^i$, before being detached and then become a wear debris. Secondly, this material turned into a wear particle is taken away from $\Omega_{3i}$ to $\Omega_{33}$. In $\Omega_{3i}$, $\varphi^i \simeq 1$. We introduce the parameter $\alpha^i(x)$, $i = 1,2$, the volume fraction of material $i$, which goes from $\Omega_{3i}$ to $\Omega_{33}$, assuming that only a part of the material in $\Omega_{3i}$ is removed into the $\Omega_{33}$ mixture. ($\alpha^i$ can also be considered as a constant parameter, equal to 1 if the last assumption cannot be made.) We can consider that the flow of wear particles goes from $\Omega_{3i}$ to $\Omega_{33}$ at the same velocity as it goes through $\Gamma_i$ ($v^i$).

We assume that the fluid is incompressible and the particles rigid once detached, and take $-V/2$ as averaged shear velocity for the particles and the fluid in the interface, treated as a homogeneous layer. This means that the two-dimensional fluid flow near the end points $x = \pm a$ of the contact interface is disregarded. Taking account of particle incomes $\alpha^i v^i \mathrm{d}x$ due to wear in a section of $\Omega_3$ between $x$ and $x + \mathrm{d}x$, the balance of mass can be written for solid particles and fluid, in the steady-state case:

$$\text{solid} \quad \alpha^i(x)v^i(x) + \frac{V}{2}\frac{\partial}{\partial x}[\overline{\varphi}^i(x)e(x)] = 0, \quad i = 1,2, \tag{4}$$

$$\text{fluid} \quad \frac{\partial}{\partial x}\left[(1 - \overline{\varphi}^1(x) - \overline{\varphi}^2(x))e(x)\right] = 0. \tag{5}$$

If there is no wear particle in the interface, its thickness is the sum of rugosities of both solids equal to $e_0$; Eq. (5) becomes $(1 - \overline{\varphi}^1(x) - \overline{\varphi}^2(x))e(x) = e_0$ on the one hand. On the other hand, the wear criterion gives the velocity $v^i$. Both relations (4) and (5) are enough to determine $\overline{\varphi} = \overline{\varphi}^1 + \overline{\varphi}^2$ needed to evaluate stresses and strains in the interface, giving by the way the interface law.

### 3.2.2. *Stresses and strains in the interface*

Normal compression and viscous shear:

$$\overline{\sigma}_{yy} = \kappa(\overline{\varphi})\overline{\varepsilon}_{yy}, \tag{6}$$

$$\overline{\sigma}_{xy} = \eta(\overline{\varphi})\overline{\dot{\varepsilon}}_{xy}. \tag{7}$$

Strains are given by the boundary conditions:

$$\overline{\varepsilon}_{yy} = \frac{u_y^1 - u_y^2}{e(\overline{\varphi})}, \tag{8}$$

$$\overline{\dot{\varepsilon}}_{xy} = \frac{\dot{u}_x^1 - \dot{u}_x^2}{e(\overline{\varphi})}. \tag{9}$$

The coefficient $\kappa(\overline{\varphi}) = [\zeta(\overline{\varphi})]^{-1} = \left[(\overline{\varphi}^1/K^1) + (\overline{\varphi}^2/K^2) + (\overline{\varphi}^f/K^f)\right]^{-1}$ corresponds to Reuss's law of strains additionality or stress homogeneity. In the case of an incompressible fluid, $\kappa(\overline{\varphi}) = [\zeta(\overline{\varphi})]^{-1} = \left[(\overline{\varphi}^1/K^1) + (\overline{\varphi}^2/K^2)\right]^{-1}$. The coefficient $\eta(\overline{\varphi}) = \eta_0[1 + 2.5\overline{\varphi}]$ designates the viscosity given by Einstein's law for the viscosity of a solid particle suspension (Landau and Lifchitz, 1971; van der Werff and de Kruif, 1989; de Kruif et al., 1985; Dragon-Louiset, 2000). In the case of mild wear, $\varphi$ remains small ($\varphi_{\max} \simeq 0.6$ for a compact assembly of circles of same radius).

The wear criterion needs to be evaluated near the boundaries $\Gamma_i$ for both sides $\Omega_i$ ($g^i$) and $\Omega_{3i}$ ($g^{3i}$). The behavior of $\Omega_1$ and $\Omega_2$ is known, so $g^i$ can easily be calculated and the behavior of $\Omega_{3i}$, for which mean values are no more appropriate, will give $g^{3i}$. In $\Omega_{3i}$, $\varphi^i \simeq 1$, $\sigma_{xx} \neq 0$ and $\sigma_{yy} \neq 0$. $\sigma_{yy}^i = \sigma_{yy}^{3i}$, $u^i = u^{3i}$. Then, we can take $\sigma_{yy}^i = \sigma_{yy}^{3i} = \kappa(\varphi^i = 1)\overline{\varepsilon}_{yy}$, $\varepsilon_{xx}^{3i} = \varepsilon_{xx}^i = u_{x,x}^i$, $\sigma_{xx}^{3i} = \kappa'(\varphi^i = 1)\varepsilon_{xx}^{3i}$.

Later on, we will employ the quantities $m[\varphi] = \eta(\varphi)/e(\varphi)$ and $k[\varphi] = \kappa(\varphi)^{-1}e(\varphi) = \zeta(\varphi)e(\varphi)$, where $\overline{\varphi}$ was noted $\varphi$ for simplicity. The following resumes the three modeling steps.

### 3.3. The interface model

(1) *wear criterion and velocity*

$$G(g^i, g^{3i}) = g^i - g^{3i} - g^{is} \geqslant 0 \quad v^i = F(\langle G(g^i, g^{3i})\rangle_+), \quad (i = 1, 2).$$

(2) *conservation of mass*

$$\begin{aligned} &\text{solid} \quad \alpha^i(x)v^i(x) + \frac{V}{2}\frac{\partial}{\partial x}[\varphi^i(x)e(x)] = 0 \quad (i = 1, 2), \\ &\text{fluid} \quad (1 - \varphi^1(x) - \varphi^2(x))e(x) = e_0. \end{aligned}$$

(3) *viscous shear and pressure*

$$\sigma_{xy}(x) = m[\varphi](x)(\dot{u}_x^1 - \dot{u}_x^2), \quad \sigma_{yy}(x)k[\varphi](x) = u_y^1(x) - u_y^2(x).$$

## 4. The wear equation for incompressible fluid

The case of an elastic half-space (solid 2) and a cylindrical rigid punch (solid 1) is examined with an incompressible fluid. Only one of the solids is worn, the half-space, it follows that there is only one kind of debris in the interface: $\overline{\varphi} = \overline{\varphi}^2$ and $1 = \overline{\varphi} + \overline{\varphi}^{\mathrm{f}}$. In order to obtain displacements, stresses and strains at the surface of the half-space, which is covered in the contact area by the interface made of fluid and wear debris, we will use the Galin's integral equations (Galin, 1953). The study of an elastic half-space submitted to contact wear due to a rigid punch was done by Galin (1976) and Galin and Goriacheva (1977). In those papers, there is nevertheless no wear debris between the punch and the half-space, therefore wear phenomena in those studies has no influence on the contact pressure and displacements. Our purpose is to take into account the presence and the influence of the third body using the interface model.

### 4.1. Problem's statement

The $x$- and $y$-axes are moving with the punch, whose symmetry axis is off-center and has abscissa $x_0$. The displacements of the punch $u^1$ denoted $u^{\mathrm{p}}$ are $u_x^{\mathrm{p}}$, with the following assumption $\dot{u}_x^{\mathrm{p}} \simeq 0$, and $u_y^{\mathrm{p}}(x) = \delta + (x - x_0)^2/(2R)$. From now on, $v^2$, $\alpha^2$, $u^2$, $\overline{\varphi}$, $\overline{\sigma}_{yy}$, $K^2$ and $\overline{\sigma}_{xy}$ are, respectively, denoted as $v$, $\alpha$, $u$, $\varphi$, $\sigma_{yy}$, $K$ and $\sigma_{xy}$ in order to simplify.

The contact area is bounded by $x = a$ and $x = -a$. Ahead of the punch, no debris is present in the interface: $\varphi(x = a) = 0$. Let us take $\alpha$ a constant parameter; the balance of mass Eq. (4) becomes $\alpha v(x) + (e_0V/2)(\partial\varphi/\partial x)(x) = 0$. By the way, once $v$ is induced by the wear criterion, $\varphi$ is given by

$$\varphi(x) = -\frac{2\alpha}{Ve_0}\int_a^x v(t)\,\mathrm{d}t, \quad x \leqslant a. \tag{10}$$

In the case of mild wear, keeping first-order terms for $\varphi$ in the model of the interface is enough. Relations between stresses and displacements described before are $\sigma_{xy} = \eta(\varphi)/e(\varphi)(\dot{u}_x^{\mathrm{p}} - \dot{u}_x)$ and $\sigma_{yy}\zeta(\varphi)e(\varphi) = u_y^{\mathrm{p}} - u_y$. The fluid being incompressible, $\kappa(\varphi) = [\zeta(\varphi)]^{-1} = [\varphi/K]^{-1}$. The layer's thickness $e(\varphi) = e_0/(1 - \varphi)$ is given by Eq. (5), thus by linearization, we denote

$$m[\varphi]=\frac{\eta(\varphi)}{e(\varphi)}=\frac{\eta_0}{e_0}(1+1.5\varphi)\quad\text{and}\quad k[\varphi]=\zeta(\varphi)e(\varphi)=\frac{e_0}{K}\varphi. \tag{11}$$

Assuming that $\dot{u}_x^{\rm p}\simeq 0$ for the punch and $\dot{u}_x=-V(\partial u_x/\partial x)\simeq -V$ for the half-space, we obtain

$$\sigma_{xy}(x)=m[\varphi](x)(\dot{u}_x^{\rm p}-\dot{u}_x)\ \Rightarrow\ \sigma_{xy}(x)=\frac{\eta_0}{e_0}[1+1.5\varphi(x)]V, \tag{12}$$

$$\sigma_{yy}(x)k[\varphi](x)=u_y^{\rm p}(x)-u_y(x)\ \Rightarrow\ \sigma_{yy}(x)\frac{e_0}{K}\varphi(x)=u_y^{\rm p}(x)-u_y(x). \tag{13}$$

Because of the assumptions made on $\dot{u}_x^{\rm p}$ and $\dot{u}_x$, Eq. (12) is similar to a plasticity threshold, which depends here on the internal variable $\varphi$ and on $V$. It is consistent with phase III of Stribeck's curve, where $\tau/p\propto\eta V/p$ i.e. $\tau\propto\eta V$.

### 4.2. The wear equation

Galin's equations (Galin, 1953) are

$$\frac{E}{2(1-v^2)}\frac{{\rm d}u_x}{{\rm d}x}(x)=\frac{(1-2v)}{2(1-v)}\sigma_{yy}(x)+\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\sigma_{xy}(s)\,{\rm d}s}{s-x}, \tag{14}$$

$$\frac{E}{2(1-v^2)}\frac{{\rm d}u_y}{{\rm d}x}(x)=-\frac{(1-2v)}{2(1-v)}\sigma_{xy}(x)+\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}(s)\,{\rm d}s}{s-x}. \tag{15}$$

Reciprocal relations can be found in Bui (1993). The principal value is denoted as **pv**, defined by

$$\mathbf{pv}\int_a^b\frac{f(s)\,{\rm d}s}{s-x}=\lim_{\epsilon\to 0}\left[\int_a^{x-\epsilon}\frac{f(s)\,{\rm d}s}{s-x}+\int_{x+\epsilon}^{b}\frac{f(s)\,{\rm d}s}{s-x}\right]. \tag{16}$$

Solutions $f$ of $\mathbf{pv}\int_a^b(f(s)\,{\rm d}s/(s-x))=g(x)$ where $g$ is a given function satisfying some regularity conditions, can be found in (Muskhelishvili, 1953, 1977). With $c_1=E/[2(1-v^2)]$ and $c_2=(1-2v)/[2(1-v)]$. By replacing $u_y$ using Eq. (13) in Eq. (15), and $\sigma_{xy}$ using Eq. (12) in both relations (14) and (15), we obtain

$$c_1\frac{{\rm d}u_x}{{\rm d}x}(x)=c_2\sigma_{yy}(x)+\frac{\eta_0 V}{e_0}\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{[1+1.5\varphi(s)]\,{\rm d}s}{s-x}, \tag{17}$$

$$c_1\frac{e_0}{K}\frac{\rm d}{{\rm d}x}\left[\sigma_{yy}(x)\varphi(x)\right]+\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}(s)\,{\rm d}s}{s-x}=c_1\frac{{\rm d}u_y^{\rm p}}{{\rm d}x}(x)+c_2\frac{\eta_0 V}{e_0}[1+1.5\varphi(x)]. \tag{18}$$

The criterion gives $v=F(g^2,g^{32})$, $\varphi$ is determined by the balance of mass equation (10) and $\sigma_{xy}$ by the viscous interface law equation (12). We have to solve Eq. (18) to obtain $\sigma_{yy}$ (where $\sigma_{yy}$ appears as well in the integral **pv** as outside this integral), $u_y$ being then given by Eq. (13) and $u_x$ by Eq. (17). Finally, the wear equation (18) for $\sigma_{yy}$ is the one to solve in order to obtain stresses, displacements and strains at the surface of the half-space. Its solution can be computed, or evaluated approximately for a given type of function (for example if $\sigma_{yy}$ is developed into a Fourier series or into Chebyshev's polynomial). Another way to solve this equation is to search asymptotic expansion solutions, with respect to a small parameter related to $\varphi(x)$. This is discussed in the following sections.

The normal stress $\sigma_{yy}$ can be computed. An iterative algorithm can determine approximated fields $\sigma$, $\varepsilon$, then calculate $v$, $\varphi$, solve the wear equation (18) and update $\varphi$, repeating this until convergence. This convergence is briefly discussed here.

Considering that the normal stress $\sigma_1$ is obtained knowing $\sigma_0$, after the evaluation of $v_0$ and $\varphi_0$, all mappings are linear except the first one $v_0 = F(\sigma_0)$, which is generally nonlinear, depending on the physics and experimental conclusions. Assuming that $F$ is a Lipschitzian function and the mapping $M$ giving the solution $\sigma_{n+1} = \sigma_0 + M(\sigma_n)$ can be proved to be a contraction for small values of $\varphi$, the problem solution is the fixed point of the nonlinear mapping: $\sigma = \sigma_0 + M(\sigma)$ (Courant and Hilbert, 1962). By the way, the convergence of the algorithm can be proved.

## 5. Asymptotic expansion solutions for a mixture with incompressible fluid

In the case of mild wear, $\varphi$ is assumed to be small. We will consider that the zero-order solution is obtained for $\varphi = 0$ (no wear occurs). Higher-order solutions are deduced from the zero-order one and other lower-order solutions. The small parameter used for the expansion can be $\omega = \sup_{|x|\leq a} |\varphi(x)|$; the asymptotic expansion fields take the form:

$$\begin{aligned}
v &= \omega v^{(1)} + \omega^2 v^{(2)} + \cdots, \\
\varphi &= \omega\varphi^{(1)} + \omega^2\varphi^{(2)} + \cdots, \\
u_x &= u_x^{(0)} + \omega u_x^{(1)} + \omega^2 u_x^{(2)} + \cdots, \\
u_y &= u_y^{(0)} + \omega u_y^{(1)} + \omega^2 u_y^{(2)} + \cdots, \\
\sigma_{xy} &= \sigma_{xy}^{(0)} + \omega\sigma_{xy}^{(1)} + \omega^2\sigma_{xy}^{(2)} + \cdots, \\
\sigma_{yy} &= \sigma_{yy}^{(0)} + \omega\sigma_{yy}^{(1)} + \omega^2\sigma_{yy}^{(2)} + \cdots
\end{aligned} \tag{19}$$

with no zero-order terms for $v$ and $\varphi$; $v^{(1)}$, $\varphi^{(1)}$ being the normalized wear rate and volume fraction, respectively. The following equations given in Section 4

$$u_y^{\mathrm{p}}(x) = \delta + \frac{(x-x_0)^2}{(2R)}, \tag{20}$$

$$v = F(g^2, g^{32}), \tag{21}$$

$$\varphi(x) = -\frac{2\alpha}{Ve_0}\int_a^x v(t)\,\mathrm{d}t, \quad x \leqslant a, \tag{22}$$

$$\sigma_{xy}(x) = \frac{\eta_0 V}{e_0}[1 + 1.5\varphi(x)], \tag{23}$$

$$\sigma_{yy}(x)\frac{e_0}{K}\varphi(x) = u_y^{\mathrm{p}}(x) - u_y(x) \tag{24}$$

will now be used to find zero-order and higher-order solutions together with the wear equation

$$c_1\frac{e_0}{K}\frac{\mathrm{d}}{\mathrm{d}x}\left[\sigma_{yy}(x)\varphi(x)\right] + \mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}(s)\mathrm{d}s}{s-x} = c_1\frac{\mathrm{d}u_y^{\mathrm{p}}}{\mathrm{d}x}(x) + c_2\frac{\eta_0 V}{e_0}[1 + 1.5\varphi(x)], \tag{25}$$

$$c_1\frac{\mathrm{d}u_x}{\mathrm{d}x}(x) = c_2\sigma_{yy}(x) + \frac{\eta_0 V}{e_0}\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{[1 + 1.5\varphi(s)]\mathrm{d}s}{s-x}. \tag{26}$$

The wear rate $v$ is function of the energy release rates $g^2$ and $g^{32}$, which depend on $\sigma_{xx}$ and $\sigma_{yy}$. Let $v^{(i)}$ at the $i$-order be obtained with only the $(i-1)$-order stresses $\sigma_{yy}^{(i-1)}$, $\sigma_{xx}^{(i-1)}$:$v^{(i)} = F[g^2, g^{32}(\sigma_{yy}^{(i-1)}, \sigma_{xx}^{(i-1)})]$. This assumption is not necessary to solve the problem and has no important consequence on its solution, but it simplifies the equations for analytical study. Notice that $\sigma_{xx}^{(i-1)}$ is obtained from $\varepsilon_{xx}^{(i-1)} = (\mathrm{d}u_x^{(i-1)}/\mathrm{d}x)$.

Introducing asymptotic expansions in Eq. (25), we obtain

$$\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\left[\sigma_{yy}^{(0)}(s)+\omega\sigma_{yy}^{(1)}(s)+\omega^2\sigma_{yy}^{(2)}(s)+\cdots\right]\mathrm{d}s}{s-x}=h_0(x)+\omega h_1(x)+\omega^2h_2(x)+\cdots \tag{27}$$

with

$$\begin{cases} h_0(x)=c_1\frac{\mathrm{d}u_y^{\mathrm{p}}}{\mathrm{d}x}(x)+c_2\frac{\eta_0 V}{e_0}, \\ h_i(x)=-c_1\frac{e_0}{K}\frac{\mathrm{d}}{\mathrm{d}x}\left[\sum_{n=0}^{i-1}\sigma_{yy}^{(n)}(x)\varphi^{(i-n)}(x)\right]+c_2\frac{1.5\eta_0 V}{e_0}\varphi^{(i)}(x) \quad \text{for } i\geqslant 1. \end{cases} \tag{28}$$

Because $u_y^{\mathrm{p}}$ is given by Eq. (20), $h_0$ is a regular function and satisfies a Hölder condition on $[-a,a]$ (Muskhelishvili, 1953). Zero-order solution can easily be determined, as shown hereafter. This is no more valid for higher orders. However, on the assumption that first-order terms for $\varphi$ provide precise enough results (Section 4.1), there is no sense in searching higher-order solutions; thus, only zero- and first-order problems are solved. Total pressure in the contact area is inferred from both zero- and first-order solutions:

$$P=P_0+\omega P_1=-\int_{-a}^{a}\left[\sigma_{yy}^{(0)}(x)+\omega\sigma_{yy}^{(1)}(x)\right]\mathrm{d}x. \tag{29}$$

### 5.1. Zero-order solution

At first, let us consider the contact without wear to determine the zero-order solution. According to Eqs. (23) and (24), respectively, $\sigma_{xy}^{(0)}=\eta_0 V/e_0$, $u_y^{(0)}(x)=u_y^{\mathrm{p}}(x)$. The stress $\sigma_{yy}^{(0)}$ is given by Eq. (25), where $\varphi=0$ and the punch displacement was replaced by Eq. (20):

$$\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}^{(0)}(s)\,\mathrm{d}s}{s-x}=\frac{c_1}{R}x+\left[\frac{c_2\eta_0 V}{e_0}-\frac{c_1x_0}{R}\right]=h_0(x). \tag{30}$$

(See Appendix A for the solution.) The consistency condition $\int_{-a}^{a}h_0(s)(a^2-s^2)^{-1/2}\,\mathrm{d}s=0$ concerning the right-hand side of Eq. (30) yields $h_0(x)=c_1x/R$ and $x_0$ is obtained by setting the bracketed term of Eq. (30) to zero. The zero-order solution $\sigma_{yy}^{(0)}$ is, therefore, the Hertzian contact pressure. We finally determine $(\mathrm{d}u_x^{(0)}/\mathrm{d}x)(x)$ using Eq. (26). The zero-order solution is given by

$$\begin{aligned} &v^{(0)}=0,\\ &\varphi^{(0)}=0,\\ &\sigma_{xy}^{(0)}=\frac{\eta_0 V}{e_0},\\ &\sigma_{yy}^{(0)}(x)=-\frac{c_1}{R}\sqrt{a^2-x^2},\\ &\frac{\mathrm{d}u_x^{(0)}}{\mathrm{d}x}(x)=-\frac{c_2}{R}\sqrt{a^2-x^2}+\frac{\eta_0 V}{\pi c_1 e_0}\ln\left(\frac{a-x}{a+x}\right),\\ &x_0=\frac{c_2}{c_1}\frac{\eta_0 VR}{e_0},\\ &u_y^{(0)}(x)=u_y^{\mathrm{p}}(x)=\delta+(x-x_0)^2/(2R). \end{aligned} \tag{31}$$

### 5.2. First-order solution

Assuming that $v^{(1)}=F[g^2,g^{32}(\sigma_{yy}^{(0)},\sigma_{xx}^{(0)})]$, $\varphi^{(1)}(x)$ is determined by Eq. (22), $\sigma_{xy}^{(1)}$ and $u_y^{(1)}$ are given by Eqs. (23) and (24), with $\sigma_{yy}^{(1)}$ solution of

$$\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}^{(1)}(s)\,\mathrm{d}s}{s-x} = -c_1\frac{e_0}{K}\frac{\mathrm{d}}{\mathrm{d}x}[\sigma_{yy}^{(0)}(x)\varphi^{(1)}(x)] + c_2\frac{1.5\eta_0 V}{e_0}\varphi^{(1)}(x) = h_1(x). \tag{32}$$

$\varphi^{(1)}(a) = 0$ but $\varphi(-a) \neq 0$ Eq. (22). The derivative $h_1$ in Eq. (32) becomes therefore singular for $x = -a$, because $\sigma_{yy}^{(0)} = -(c_1/R)\sqrt{a^2 - x^2}$. (For the criterion presented before and in the case of a linear or polynomial function $F$ Eq. (21), taking into account that $v$ depends on $\sigma_{yy}$ squared, $\varphi^{(1)}(x)$ (22) is a polynomial with no singularity.)

It implies that $h_1$ does not satisfy a Hölder condition in $x = -a$. Let us express $h_1$ as a sum of four terms; its regular part $h_1^{\mathrm{r}}$ for $x \in [-a, a]$, its singular part $h_1^{\mathrm{s}}$, $\omega C(a + x)$ and $-\omega C(a + x)$ with $C$ a constant to determine: $h_1(x) = h_1^{\mathrm{r}}(x) + h_1^{\mathrm{s}}(x) + \omega C(a + x) - \omega C(a + x)$. Instead of solving Eq. (32), we will move forward $h_1^{\mathrm{s}}(x) - \omega C(a + x)$ to the next order equation. Thus, Eq. (32) is replaced by

$$\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\sigma_{yy}^{(1)}(s)\,\mathrm{d}s}{s-x} = h_1^{\mathrm{r}}(x) + \omega C(a + x), \tag{33}$$

where the right-hand member satisfies a Hölder condition on $[-a, a]$. The term $(1/\omega)h_1^{\mathrm{s}}(x) - C(a + x)$ is added to $h_2(x)$ for second-order equation; if further orders solutions are searched, a similar regularization is needed for each step. The constant $C$ is given by the consistency condition

$$\int_{-a}^{a}\left[h_1^{\mathrm{r}}(s) + \omega C(a + s)\right](a^2 - s^2)^{-1/2}\,\mathrm{d}s = 0. \tag{34}$$

The first-order solution is then

$$\begin{aligned}
&v^{(1)}(x) = F[g^2, g^{32}(\sigma_{yy}^{(0)}, \sigma_{xx}^{(0)})],\\
&\varphi^{(1)}(x) = -\frac{2\alpha}{Ve_0}\int_a^x v^{(1)}(t)\,\mathrm{d}t,\\
&\sigma_{xy}^{(1)}(x) = \frac{1.5\eta_0 V}{e_0}\varphi^{(1)}(x),\\
&C = \text{given by Eq. (34)},\\
&\sigma_{yy}^{(1)}(x) = \text{solution of Eq. (33)},\\
&\frac{\mathrm{d}u_x^{(1)}}{\mathrm{d}x}(x) = -\frac{c_2}{c_1}\sigma_{yy}^{(1)}(x) + \frac{1.5\eta_0 V}{c_1 e_0}\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{\varphi^{(1)}(s)\,\mathrm{d}s}{s-x},\\
&u_y^{(1)}(x) = -\frac{e_0}{K}\sigma_{yy}^{(0)}(x)\varphi^{(1)}(x).
\end{aligned} \tag{35}$$

### 5.3. Analytical example

The first-order term $\sigma_{yy}^{(1)}$ can be determined analytically for a simplified wear criterion. Taking for example $G(g^2, g^{32}) = \Theta(\sigma_{yy})^2$, and $v = \beta\langle G(g^2, g^{32})\rangle_+$, the first-order velocity is given by $v^{(1)} = \beta\Theta(\sigma_{yy}^{(0)})^2$, according to the assumptions made above. Using Eq. (35), the volume fraction $\varphi^{(1)}$, the constant $C$ and the first-order normal stress $\sigma_{yy}^{(1)}$ are calculated. The parameter $\alpha\beta\Theta$ is expressed as $\chi$. The constant $C$ is evaluated so that $h_1^{\mathrm{r}}(x) + \omega C(a + x)$ satisfies the consistency condition Eq. (34), which implies $C = (\chi c_1^2/R^2)2a^2\left[(4/3\pi)(c_1^2/KVR) - (c_2\eta_0/e_0^2)\right]$. Finally, using the reciprocal formula (A.7) (Appendix A), it follows from Eq. (33) that

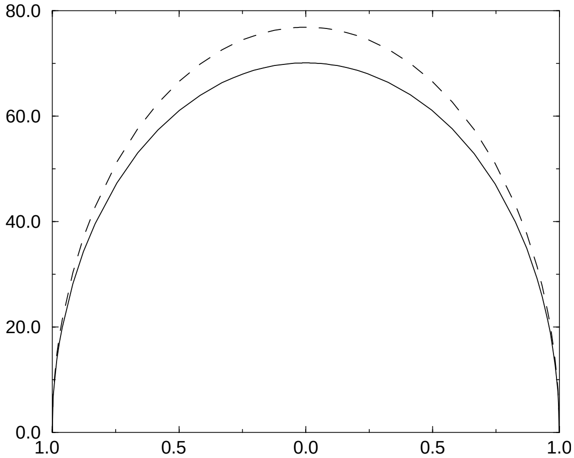


Fig. 4. $\sigma_{yy}^{(0)}$ (- - -) and $\sigma_{yy} = \sigma_{yy}^{(0)} + \omega\sigma_{yy}^{(1)}$.

$$\sigma_{yy}^{(1)} = \sigma_{yy}^{(0)} \frac{\chi c_1}{R} \left\{ -\frac{2}{\pi} \frac{c_1^2}{KVR} \left[ (a^2 - x^2) \ln \left| \frac{a-x}{a+x} \right| - 2ax \right] + \frac{c_2 \eta_0}{e_0^2} \left[ x^2 - \frac{5}{2} a^2 \right] + 2a^2 \left[ \frac{4}{3\pi} \frac{c_1^2}{KVR} - \frac{c_2 \eta_0}{e_0^2} \right] \right\}. \tag{36}$$

In Fig. 4 are drawn $\sigma_{yy}^{(0)}$ as a dashed line and $\sigma_{yy} = \sigma_{yy}^{(0)} + \omega\sigma_{yy}^{(1)}$ as a solid line (with negative units). (The following parameters was chosen: $E = 2 \times 10^{-11}$, $\nu = 0.34$, $\eta_0 = 3.5 \times 10^{-4}$, $e_0 = 1 \times 10^{-6}$, $R = 0.02$, $V = 1$, $\omega = 0.1$, $\chi = 30 \times 10^{-12}$.) The first-order normal stress $\omega\sigma_{yy}^{(1)}$ appears as a correction for $\sigma_{yy}^{(0)}$ corresponding to the influence of the wear process. For a fair estimate of this correction, some parameters set here to unity (e.g. $\beta$ necessary to evaluate the wear velocity) must be experimentally determined, as material properties.

## 6. Conclusion

A wear criterion derived from the second law of thermodynamics is proposed in this paper, taking into account the mass fluxes due to the production of the wear debris. Specific energy release rates arise in this analysis and can be interpreted as the energy dissipated during the process of asperities cracks. The wear velocities of the contacting bodies are inferred from the criterion. This global study is coupled to the interface law which describes, in the case of mild wear, the average behavior of this complex area made of damaged subsurfaces and third body. The volume fraction of particles in this interface is introduced as the internal parameter. An equation of mass conservation completes the study, providing a relation between the internal parameter and the wear velocity deduced from the wear criterion.

This model can be applied to two contacting structures losing material, and solved by the finite element method, but also, as it is proposed in this article, by integral equations. The problem of an elastic half space and a rigid punch with their interface leads to a single wear equation to solve. Its solution, the normal stresses in the contact area, can be determined as an asymptotic expansion with respect to the small parameter: the volume fraction of debris. An analytical example is proposed, for a simplified wear criterion.

The loss of material and the changes of surface geometry can be evaluated easily with this model, where the wear velocity and the volume fraction of debris are the essential parameters. An important quantity in tribology is the friction coefficient, which sometimes varies making modeling difficult; it can be inferred here from the shear and the normal stress in the interface and is not a data.

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## Appendix A

Let us solve the Hilbert equation (A.1), where $f$ is the unknown bounded function at $x = \pm a$ and $g$ is a given function, satisfying a Hölder condition for $x \in [-a, a]$:

$$g(x) = \mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{f(t)\,\mathrm{d}t}{t-x}. \tag{A.1}$$

As will be seen by using the reasoning presented below, whose details can be found in (Muskhelishvili, 1953), $f$ can be determined by a reciprocal relation. (See also Bui (1977) and Muskhelishvili (1953) for the complete mathematical background.)

Introducing the holomorphic function $\Phi(z) = \mathbf{pv}(1/2i\pi)\int_{-a}^{a}(f(t)\,\mathrm{d}t/t-z)$ vanishing at infinity, we can write the following Plemelj formulae valid for $x \in [-a, a]$

$$\Phi^{+}(x) - \Phi^{-}(x) = f(x), \tag{A.2}$$

$$\Phi^{+}(x) + \Phi^{-}(x) = \frac{g(x)}{i} = \mathbf{pv}\frac{1}{\pi i}\int_{-a}^{a}\frac{f(t)\,\mathrm{d}t}{t-x}, \tag{A.3}$$

where $\Phi^{+}(x)$ is the value of $\Phi(x)$ on the $y > 0$ side of the line $x \in [-a, a]$ and $\Phi^{-}(x)$ is its value on the $y < 0$ side. Let $Z$ be a second holomorphic function defined by $Z(z) = (a-z)^{1/2}(a+z)^{1/2}$. It can easily be noted that $Z$ satisfies $Z^{+}(x) = -Z^{-}(x)$, thus dividing the second Plemelj formula (A.3) by $Z^{+}(x)$, and similarly the first one Eq. (A.2), we obtain,

$$\frac{\Phi^{+}(x)}{Z^{+}(x)} - \frac{\Phi^{-}(x)}{Z^{-}(x)} = \frac{g(x)}{iZ^{+}(x)}, \tag{A.4}$$

$$\frac{\Phi^{+}(x)}{Z^{+}(x)} + \frac{\Phi^{-}(x)}{Z^{-}(x)} = \frac{f(x)}{Z^{+}(x)}. \tag{A.5}$$

Eqs. (A.4) and (A.5) are the two Plemelj formulae where $f(x)/Z^{+}(x) = \mathbf{pv}\frac{1}{\pi i}\int_{-a}^{a}(g(t)/iZ^{+}(t))(\mathrm{d}t/(t-x))$, on condition that $\Phi(z)/Z(z)$ vanishes at infinity, with $\Phi(z)/Z(z) = \mathbf{pv}[1/(2i\pi)]\int_{-a}^{a}[g(t)/(iZ^{+}(t))][\mathrm{d}t/(t-z)]$. (for $x \in [-a, a]$, $Z^{+}(x) = (a^2 - x^2)^{1/2}$). Because $\lim_{z\to\infty}\Phi(z) = 0$ and $\lim_{z\to\infty} Z(z) = -i|z|$, this condition is satisfied, provided that

$$\int_{-a}^{a}\frac{g(t)\,\mathrm{d}t}{(a^2-t^2)^{1/2}} = 0. \tag{A.6}$$

Under this condition, $f$ is given by

$$f(x) = -(a^2-x^2)^{1/2}\mathbf{pv}\frac{1}{\pi}\int_{-a}^{a}\frac{g(t)\,\mathrm{d}t}{(a^2-t^2)^{1/2}(t-x)}. \tag{A.7}$$

In Section 5.1, the solution to Eq. (30) can be obtained knowing that $\mathbf{pv}(1/\pi)\int_{-1}^{1}(t\mathrm{d}t/(\sqrt{1-t^2}(t-x))) = 1$ for $x \in [-1, 1]$, or using expansion of $h_0(x)/(a^2-x^2)^{1/2}$, where $h_0$ is polynomial (Muskhelishvili, 1953).

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